Interpretation and identification of path-specific effects in CT-VAR(1) models

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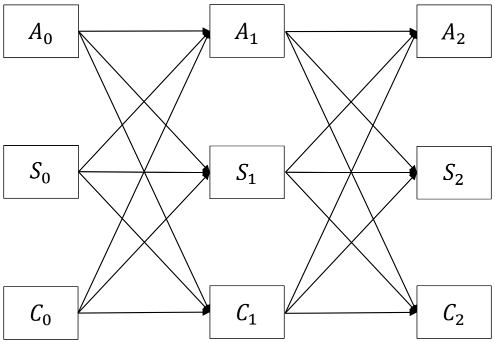
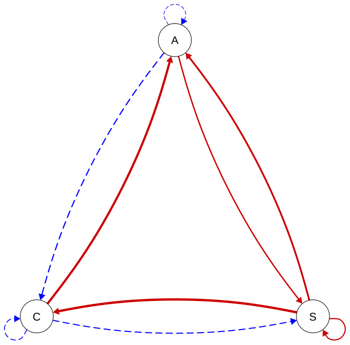
Context: Discrete-Time VAR(1) Model

$$\boldsymbol{Y}_{\tau} = \boldsymbol{\Phi}\boldsymbol{Y}_{\tau-1} + \boldsymbol{\epsilon}_{\tau}$$





Summary I

- VAR(1) often applied in ESM settings in psychology
- Interpretation of $\mathbf{\Phi}$ as **Direct Causal Effects**

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- Interpretation of $\mathbf{\Phi}$ as **Direct Causal Effects**
- Assumptions
  - Evenly spaced observations
  - Linearity of relationships
  - Stable structure over time-window of observations
  - No unobserved confounders

Summary II

- Discrete-Time VAR(1) - misleading conclusions regarding **causal structure** *even in ideal settings*

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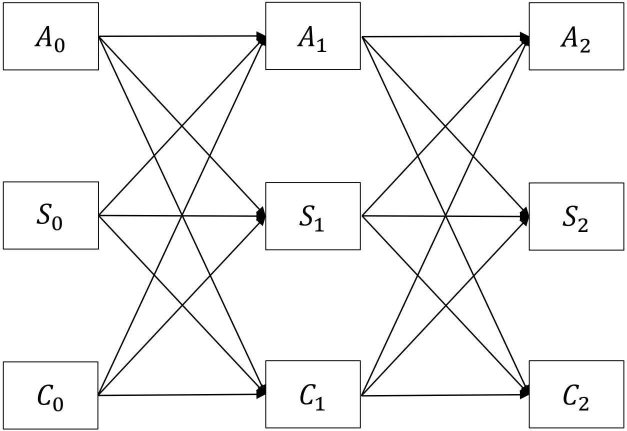
  - Alternative calculations of direct, indirect and total effects in a mediation context (Aalen et al. 2008; Deboeck & Preacher, 2015)

Summary II

- Discrete-Time VAR(1) - misleading conclusions regarding **causal structure** *even in ideal settings*
- The Continuous-Time VAR(1) model is an appealing alternative
  - Suggested by many authors, notably Voelkle, Oud and colleagues (2012), and Boker (2002)
  - Alternative calculations of direct, indirect and total effects in a mediation context (Aalen et al. 2008; Deboeck & Preacher, 2015)
- Defining these path effects in terms of **hypothetical experiments** clarifies what our target of inference is (Rubin, Pearl, Robins and others)
  - DT and CT "direct" effects describe different interventions

Path-specific Effects

$$\boldsymbol{Y}_\tau = \boldsymbol{\Phi} \boldsymbol{Y}_{\tau-1} + \boldsymbol{\epsilon}_\tau$$



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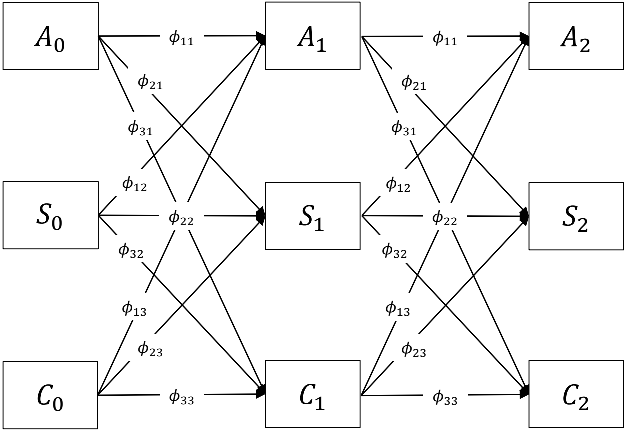

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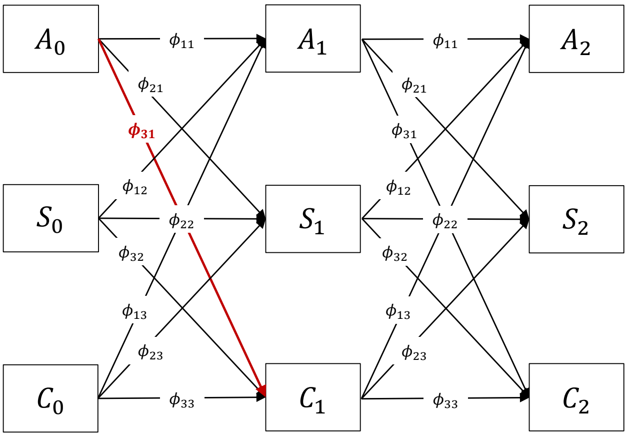

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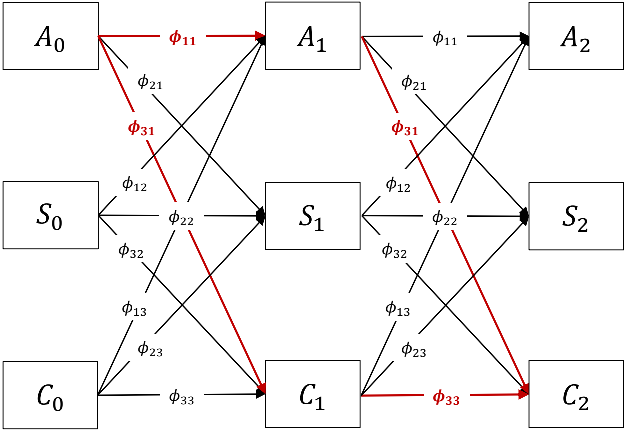

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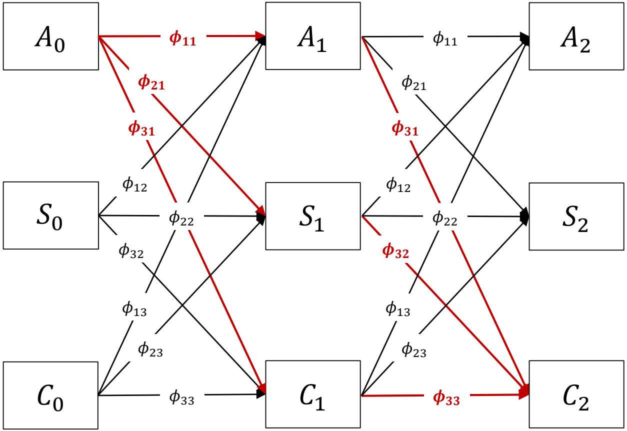

The role of time-interval (cf. Gollob & Reichardt, 1987; Cole & Maxwell, 2003)

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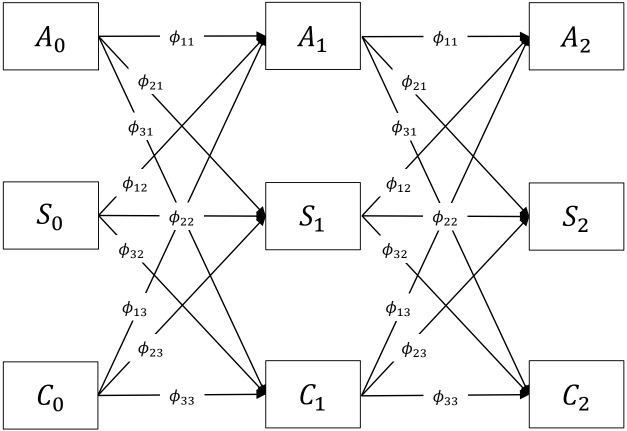

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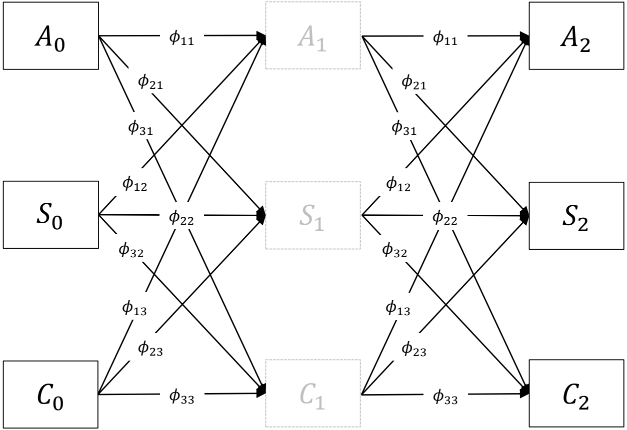
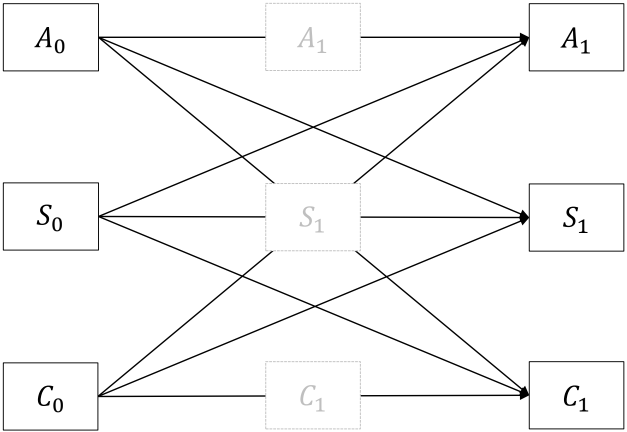


The role of time-interval (cf. Gollob & Reichardt, 1987; Cole & Maxwell, 2003)

$$\boldsymbol{Y}_\tau = \boldsymbol{\Phi}(\Delta t = 2)\boldsymbol{Y}_{\tau-1} + \boldsymbol{\epsilon}_\tau$$



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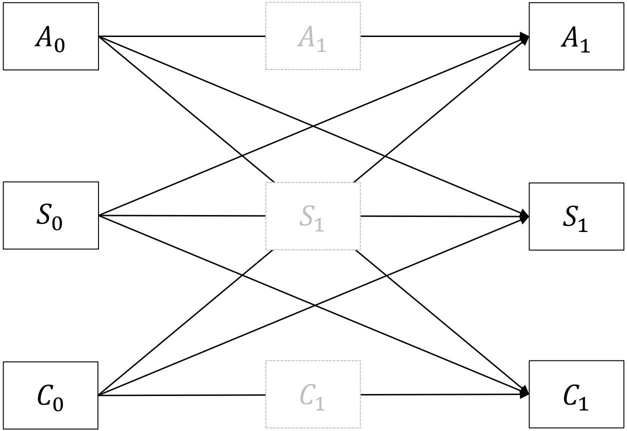

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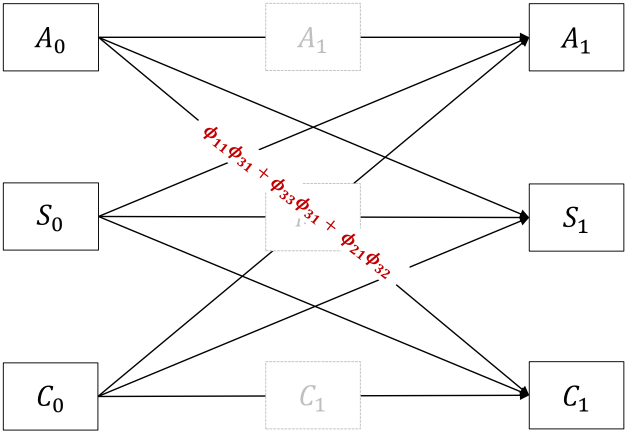
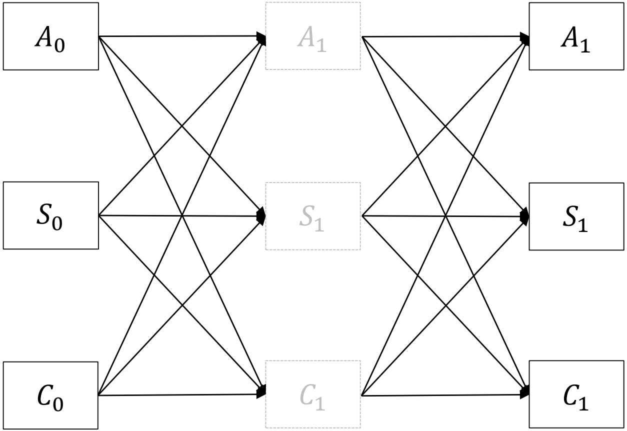


From discrete-time to continuous-time



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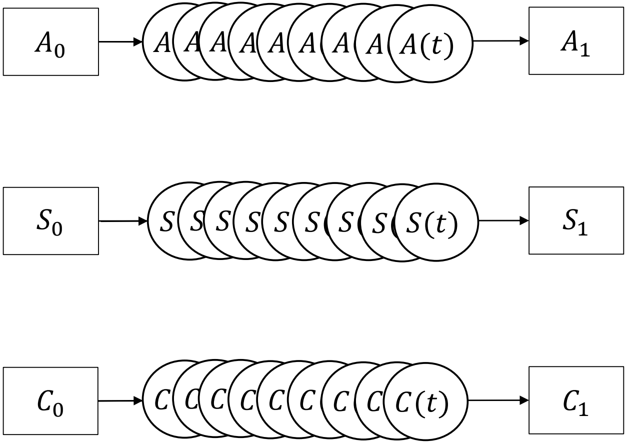

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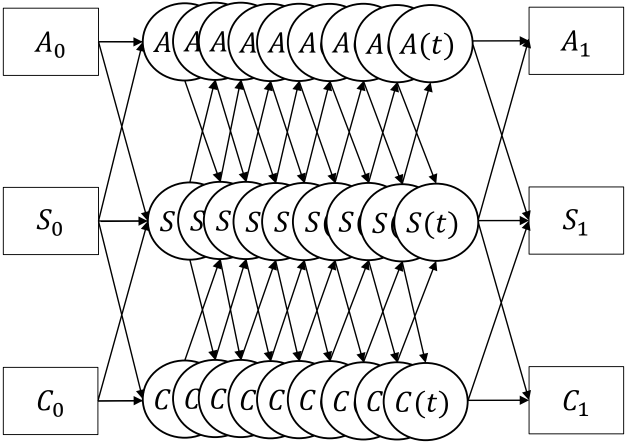
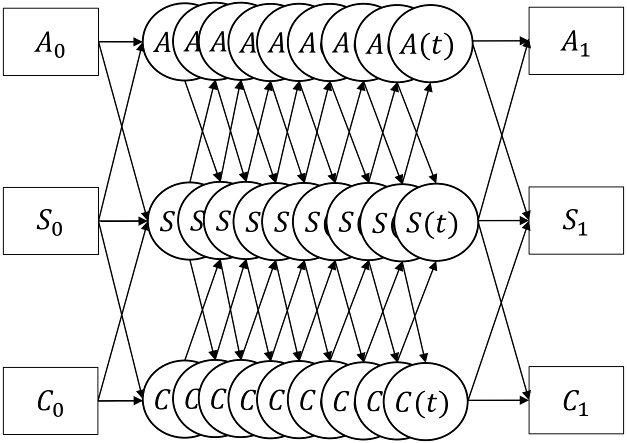


The Continuous-Time VAR(1) Model

$$\boldsymbol{Y}_\tau = \boldsymbol{e}^{\boldsymbol{A}\Delta t}\boldsymbol{Y}_{\tau-1} + \boldsymbol{\epsilon}_\tau$$



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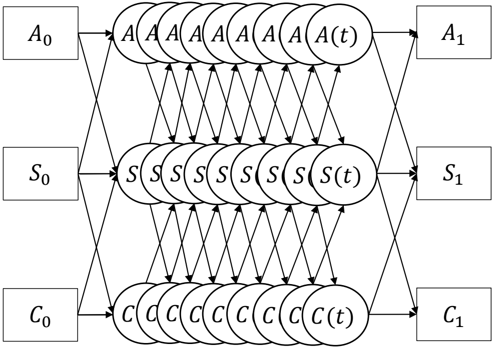

Assuming that processes:

1. take on some value at all points in time
2. exert influence on one another at all points in time
3. are smooth and differentiable

See Boker (2002) amongst others

CT-VAR(1) and Paths (Debeock & Preacher, 2015; Aalen et al. 2008)

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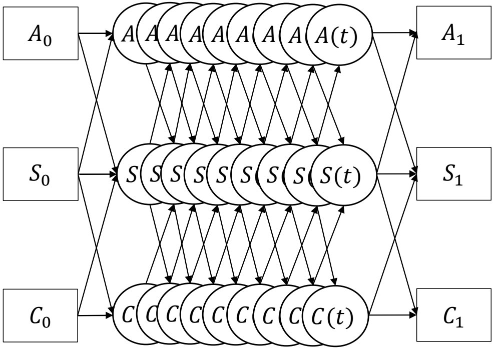

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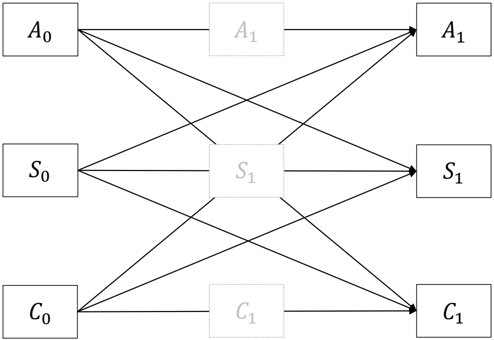

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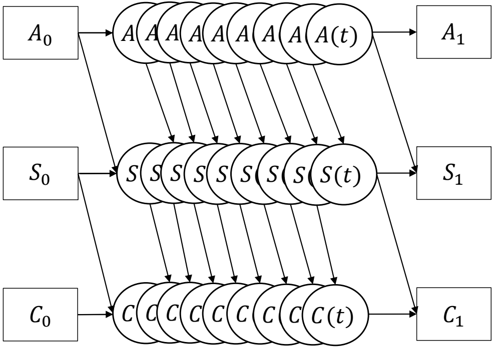

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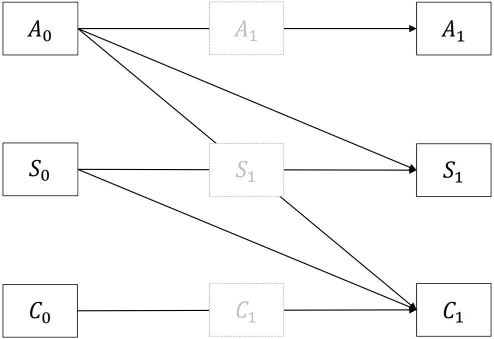
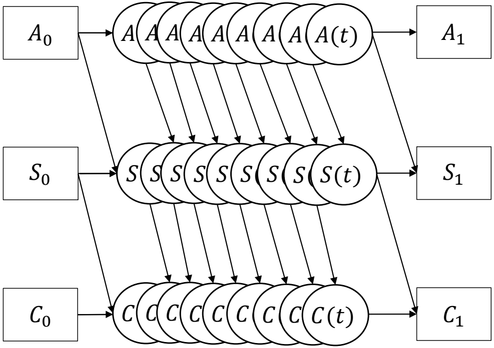
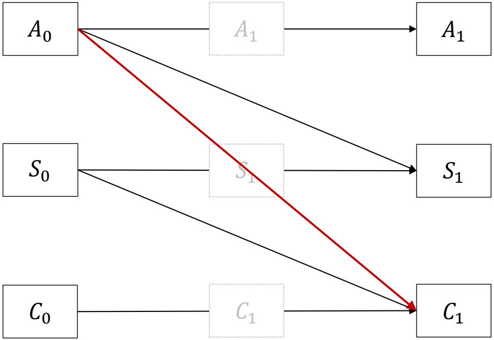


CT-VAR(1) and Paths (Debeock & Preacher, 2015; Aalen et al. 2008)

DT "Direct Effect" I $= \phi_{31}(\Delta t = 2)$





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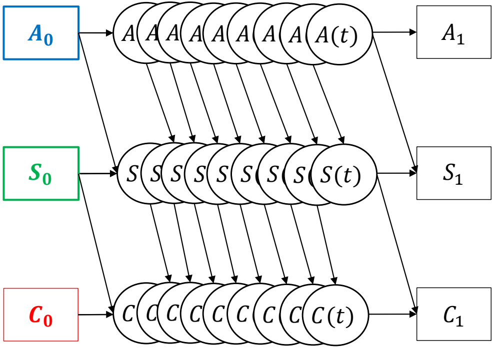

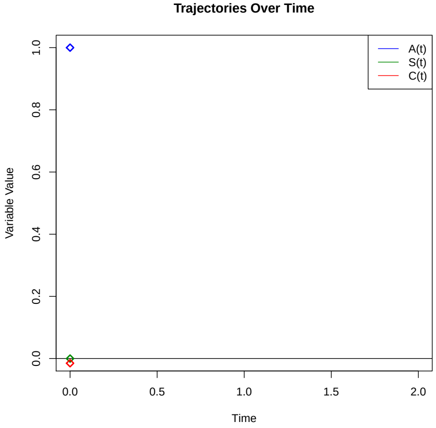

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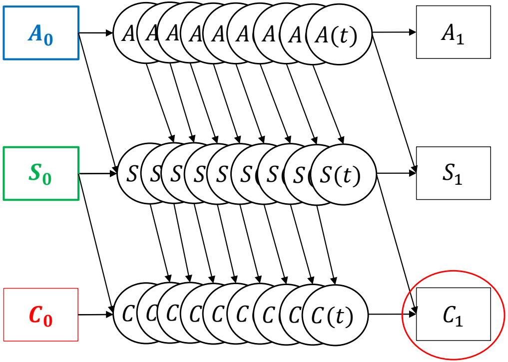
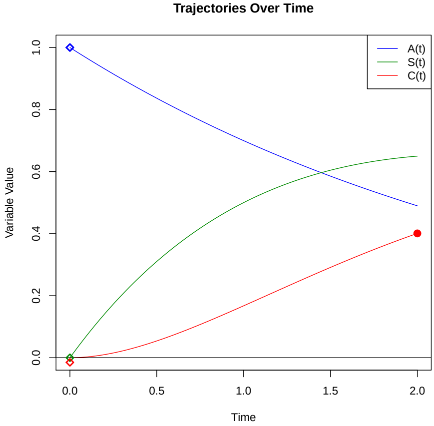
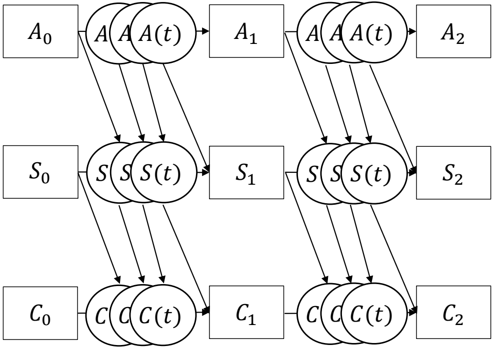
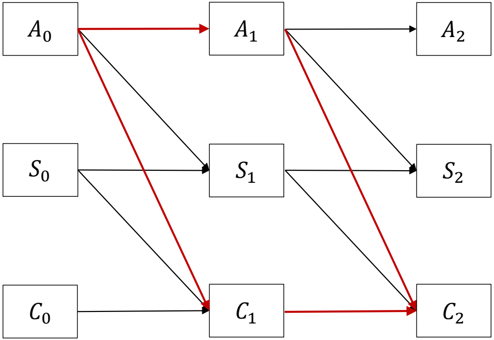




Direct Effects and Interventions

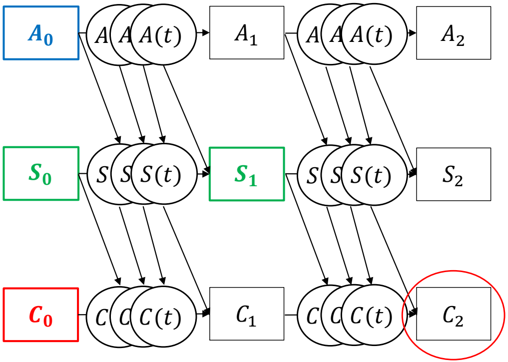
$$\text{DT "Direct Effect" II} = \phi_{11}\phi_{31} + \phi_{31}\phi_{33} \qquad [(\Delta t = 1)]$$

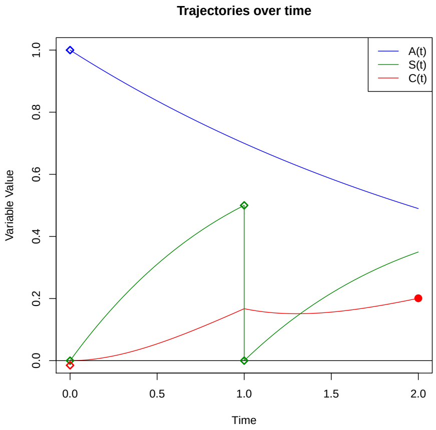




Direct Effects and Interventions

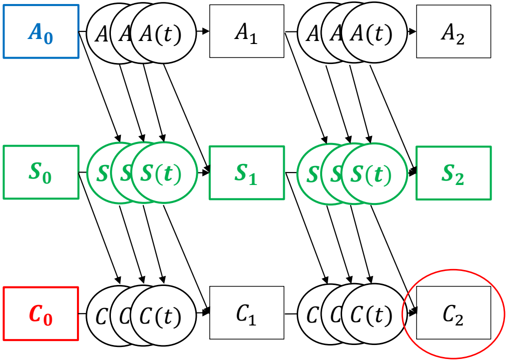
$$\text{DT "Direct Effect" II} = \phi_{11}\phi_{31} + \phi_{31}\phi_{33} \quad [(\Delta t = 1)]$$

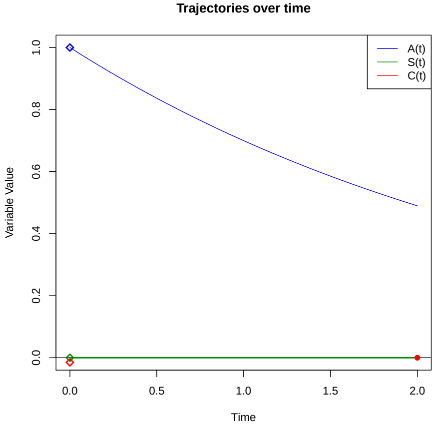




Direct Effects and Interventions

CT "Direct Effect" $= 0$





Conclusion

- Psychological processes are likely to evolve continuously over time
- "Direct effects" in DT VAR(1) models reflect specific interventions on acute values of the mediator
  - Generally doesn't match up with substantive interpretation - e.g. independent flow of information in networks
- The causal idea that best matches this is the interval intervention
  - We have shown equivalency between this interventionist definition and path-tracing approaches (Deboeck & Preacher, 2015)
  - Calculation of direct effects generalises beyond simple mediation model

In progress

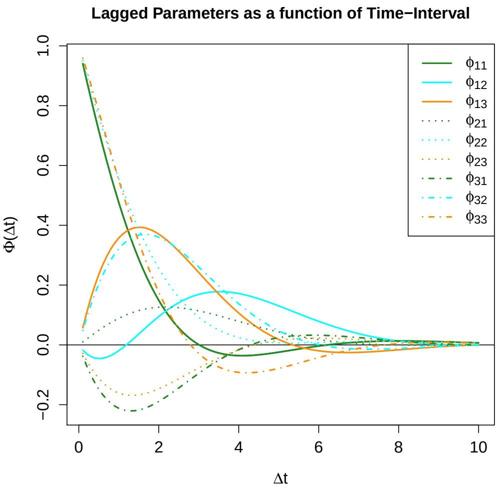
- Indirect effects less straightforward - requires variable-splitting notion - Robins (2003)
  - Not all path specific effects identifiable - "recanting witness"; Avin, Shpitser & Pearl (2005)
- Centrality measures based on CT dynamical networks

Get in Touch

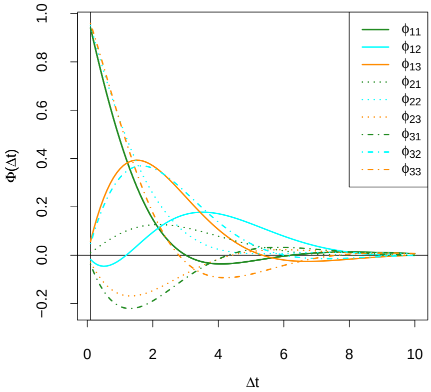
- http://dml.sites.uu.nl/
- o.ryan@uu.nl

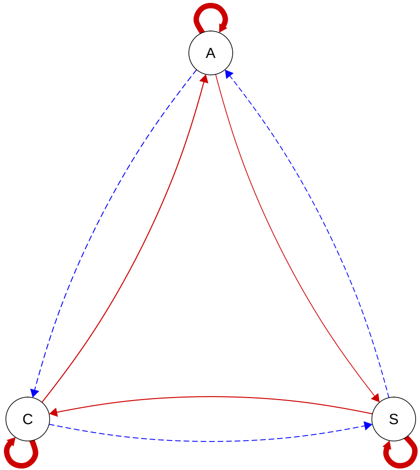
Time-interval dependency of VAR estimates

$$e^{\boldsymbol{A}\Delta t} = \boldsymbol{\Phi}(\Delta t)$$

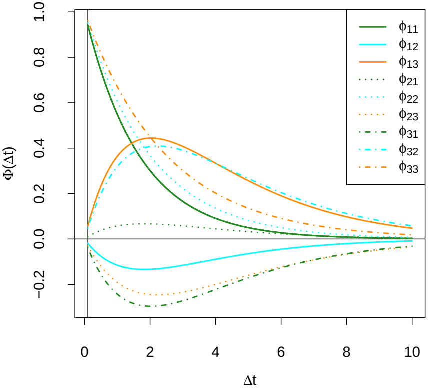


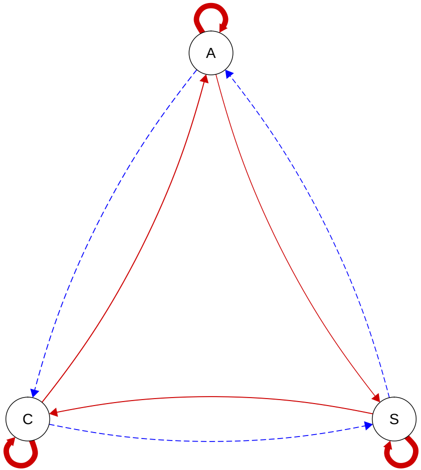
Network structure as a function of lag





CT Direct Effect Network structure as a function of lag





Estimation

- ctsem - Driver, Voelkle, Oud
- GLLA and LDE through OpenMx - Boker and colleagues
- BHOU - Oravecz and colleagues
- Indirect estimation (using DSEM in Mplus)*
- Extended Multi-level CT models - Rebecca Kuiper *

Continuous Time Model

First-Order Stochastic Differential Equation

$$\frac{d\boldsymbol{Y}(t)}{dt} = \boldsymbol{A}(\boldsymbol{Y}(t) - \boldsymbol{\mu}) + \gamma \frac{d\boldsymbol{W}(t)}{dt}$$

CT VAR(1) Model

$$\boldsymbol{Y}(t) = \boldsymbol{e}^{\boldsymbol{A}\Delta t}\boldsymbol{Y}(t - \Delta t) + \boldsymbol{w}(\Delta t)$$

Numerical Example Network

$$\boldsymbol{A} = \begin{bmatrix} -6 & -.2 & .6 \\ .1 & -.5 & -.3 \\ -.4 & .5 & -.4 \end{bmatrix}$$

Direct effect example

$$\boldsymbol{A} = \begin{bmatrix} -.357 & 0 & 0 \\ .771 & -.511 & 0 \\ 0 & .729 & -.693 \end{bmatrix}$$

Proof equivalence of path and variable setting

Deboeck & Preacher suggest finding direct effects by disabling paths in the $v \times v$ drift matrix $\boldsymbol{A}$ before applying the matrix exponential term.

Take $\boldsymbol{S}$ to be an intervention matrix; equivalent to an identity matrix with one diagonal element set to zero. E.g., if we are interested in an intervention on M, let $\boldsymbol{S} = diag(1, 0, 1)$.
$\boldsymbol{S}$ is nilpotent, thus $\boldsymbol{S}.\boldsymbol{S} = \boldsymbol{S}$

Setting the initial end final values of M in the interval to zero, this definition of the direct effect is

$$\boldsymbol{S}.\boldsymbol{e}^{\boldsymbol{S}.\boldsymbol{A}.\boldsymbol{S}\Delta t}.\boldsymbol{S}$$

Proof 1

It suffices to show that

$$\boldsymbol{S}.e^{\boldsymbol{S}.\boldsymbol{A}.\boldsymbol{S}\Delta t}.\boldsymbol{S} = \lim_{k\to\infty}(\boldsymbol{S}.e^{\boldsymbol{A}\Delta t/k}.\boldsymbol{S})^k \tag{1}$$

$$\boldsymbol{S}.e^{\boldsymbol{S}.\boldsymbol{A}.\boldsymbol{S}\Delta t}.\boldsymbol{S}$$

$$\boldsymbol{S}.\boldsymbol{I}.\boldsymbol{S} + \boldsymbol{S}.\boldsymbol{A}.\boldsymbol{S}\Delta t + \frac{\boldsymbol{S}.\boldsymbol{A}.\boldsymbol{S}^2(\Delta t)^2}{2!} + \dots$$

$$\lim_{n\to\infty}(\boldsymbol{S}.\boldsymbol{I}.\boldsymbol{S} + \frac{\boldsymbol{S}.\boldsymbol{A}.\boldsymbol{S}\Delta t}{n})^n$$

$$\lim_{k\to\infty}(\boldsymbol{S}.e^{\boldsymbol{A}\Delta t/k}.\boldsymbol{S})^k$$

As $k \to \infty$

$$e^{\boldsymbol{A}\Delta t/k} \to \boldsymbol{I} + \frac{\boldsymbol{A}\Delta t}{k}$$

$$\lim_{k\to\infty}(\boldsymbol{S}(\boldsymbol{I} + \frac{\boldsymbol{A}\Delta t}{k})\boldsymbol{S})^k$$

$$\lim_{n\to\infty}(\boldsymbol{S}.\boldsymbol{I}.\boldsymbol{S} + \frac{\boldsymbol{S}.\boldsymbol{A}.\boldsymbol{S}\Delta t}{n})^n$$